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On the Rate of Explosion for Infinite Energy Solutions of the Spatially Homogeneous Boltzmann Equation

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Abstract Let μ_0 be a probability measure on $\mathbb{R}^3$ representing an initial velocity distribution for the spatially homogeneous Boltzmann equation for pseudo Maxwellian molecules. As long as the initial energy is finite, the solution μ_t will tend to a Maxwellian limit. We show here that if $\int_{\mathbb{R}^3} |v|^2\mu_0(\mathrm{d}v) = \infty$, then instead, all of the mass "explodes to infinity" at a rate governed by the tail behavior of μ_0. Specifically, for $L0$, define

$$\eta_L = \int_{|v|\le L} |v|^2 \mathrm{d}\mu_0(v).$$

Let B_R denote the centered ball of radius R. Then for every R,

$$\lim_{t\to\infty} \int_{B_R} \mathrm{d}\mu_t(v) = 0.$$

The explicit rate is estimated in terms of the rate of divergence of η_L. For example, if $\eta_L \ge \text{Const.}L^s$, some $s > 0$, $\int_{B_R} \mathrm{d}\mu_t(v)$ is bounded by a multiple of $e^{-[\kappa 3s/(10+9s)]t}$, where κ is the absolute value of the spectral gap in the linearized collision operator. Note that in this case, letting B_t denote the ball of radius e^{rt} for any $r < \kappa s/(10+9s)$, we still have $\lim_{t\to\infty} \int_{B_t} \mathrm{d}\mu_t(v) = 0$.

This result shows in particular that the necessary and sufficient condition for $\lim_{t\to\infty} \mu_t$ to exist is that the initial data have finite energy. While the "explosion" of the mass towards infinity in the case of infinite energy may seem to be intuitively clear, there seems not to

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have been any proof, even without the rate information that our proof provides, apart from an analogous result, due to the authors, concerning the Kac equation. A class of infinite energy eternal solutions of the Boltzmann equation have been studied recently by Bobylev and Cercignani. Our rate information is shown here to provide a limit on the tails of such eternal solutions.

1 Introduction

We introduce our subject under the usual hypothesis that the initial velocity distribution μ_0 is absolutely continuous with respect to f, though this shall be relaxed later on. The Cauchy problem for the spatially homogeneous Boltzmann equation on $\mathbb{R}^3$ for pseudo-Maxwelian molecules has the form

$$\frac{\partial}{\partial t} f(v,t) = \mathcal{Q} f(v,t) \quad \text{with } f(v,0) = f_0(v). \tag{1.1}$$

Here, the collision kernel $\mathcal{Q}$ is a bilinear map from $L^1(\mathbb{R}^3) \times L^1(\mathbb{R}^3)$ to $L^1(\mathbb{R}^3)$ given by

$$\mathcal{Q}(f,g)(v) = \int_{\mathbb{R}^3} \left(\int_{S^2} (f(v_*)g(w_*) - f(v)g(w))\, B(\cos(\theta))\mathrm{d}\sigma \right) \mathrm{d}w \tag{1.2}$$

where

$$v_* = \frac{v+w}{2} + \frac{|v-w|}{2}\sigma, \qquad w_* = \frac{v+w}{2} - \frac{|v-w|}{2}\sigma$$

is a parameterization of the energy and momentum conserving collisions, through the unit vector σ, which ranges over S^2, the unit sphere in $\mathbb{R}^3$. In (1.2), $\mathrm{d}\sigma$ is the uniform probability measure on S^2, and θ denotes the angle between σ and the relative velocity, $v - w$.

The positive function B determines the relative likelihood of the various possible collision outcomes as parameterized by σ. See [10] for further background. We suppose that $\int_{S^2} B(\cos(\theta))\mathrm{d}\sigma < \infty$. Under this assumption, it is natural to normalize the time scale so that $\int_{S^2} B(\cos(\theta))\mathrm{d}\sigma = 1$.

One may then separate the collision kernel into the gain and loss terms, $\mathcal{Q}(f,g) = \mathcal{Q}^+(f,g) - \mathcal{Q}^-(f,g)$, where

$$\mathcal{Q}^+(f,g) = \int_{S^2} \int_{\mathbb{R}^3} f(v_*)g(w_*)B(\cos(\theta))\mathrm{d}w\mathrm{d}\sigma \tag{1.3}$$

and

$$\mathcal{Q}^-(f,g) = f. \tag{1.4}$$

The *Wild convolution* $f \circ g$ of two probability densities f and g on $\mathbb{R}^3$ is defined by

$$f \circ g(v) = \mathcal{Q}^+(f,g).$$

This permits (1.1) to be written in the from

$$\frac{\partial}{\partial t} f(v,t) = f \circ f(v,t) - f(v,t). \tag{1.5}$$

As Wild proved [15], the solution of (1.5) with initial data f_0 is given by

$$f(v,t) = \sum_{n=1}^{\infty} e^{-t}(1-e^{-t})^{n-1} Q_n^+(f_0)(v) \tag{1.6}$$

where $Q_n^+(f_0)$ is a recursively defined average over all the n-fold Wild convolutions of f_0: By definition, $Q_1^+(f_0) = f_0$, and then for $n \geq 1$,

$$Q_n^+(f_0) = \frac{1}{n-1} \sum_{j=1}^{n-1} Q_{n-j}^+(f_0) \circ Q_j^+(f_0). \tag{1.7}$$

Just as the ordinary convolution can be extended from L^1 to measures, the Wild convolution can also be defined for any two probability measures μ and ν. One approach to this is explained in [9], and another can be given in terms of the Fourier transform identities introduced in Sect. 2. In any case, $Q_n^+(\mu_0)$ is well defined for all probability measures μ_0 on $\mathbb{R}^3$, and then

$$\mu_t = \sum_{n=1}^{\infty} e^{-t}(1-e^{-t})^{n-1} Q_n^+(\mu_0) \tag{1.8}$$

gives the measure valued solution μ_t of (1.5) with initial data μ_0.

Our main result concerns the behavior of these solutions when the initial energy is infinite.

Theorem 1.1 *Let μ_0 be a probability measure on $\mathbb{R}^3$ such that $\int_{\mathbb{R}^3} |v|^2 \mathrm{d}\mu(v) = \infty$. Let η_L be defined by*

$$\eta_L = \int_{|v|\leq L} |v|^2 \mathrm{d}\mu(v). \tag{1.9}$$

Let B_R be the centered ball of radius R, and let μ_t be the solution of the Cauchy problem for μ_0 given by (1.8). Then for every $R > 0$

$$\lim_{t\to\infty} \int_{B_R} \mathrm{d}\mu_t = 0. \tag{1.10}$$

The rate of convergence can be explicitly estimated in terms of the rate at which η_L diverges with increasing L. For example, if $\eta_L \geq \mathrm{Const}\cdot L^s$, then

$$\int_{B_R} \mathrm{d}\mu_t \leq \mathrm{Const}\cdot R^3 e^{-[3\kappa s/(10+9s)]t},$$

where

$$\kappa = 1 - \int_{S^2} [\cos^4(\theta/2) + \sin^4(\theta/2)] B(\cos(\theta)) \mathrm{d}\sigma, \tag{1.11}$$

which happens to be the magnitude of the spectral gap for the linearized collision operator.

The proof of the theorem will yield rate information in the general case in which η_L may diverge arbitrarily slowly. Then however, as one might expect, the convergence in (1.10) can be arbitrarily slow as well. The bound on this convergence is simplest when $\eta_L \geq \mathrm{Const}\cdot L^s$.

Note that in this case, $\lim_{t\to\infty}\int_{B_t} \mathrm{d}\mu_t = 0$ where B_t is the ball or radius e^{rt}, and r is any number with $r < \kappa s/(5+9s)$.

Clearly, whenever (1.10) is true, it is impossible for μ_t to converge, as t tends to infinity, to any limiting probability distribution on $\mathbb{R}^3$, even weakly in the topology generated by $C_b(\mathbb{R}^3)$. Since whenever the initial energy is finite, μ_t does converge to a Maxwellian, even strongly in $L^1(\mathbb{R}^3)$, we see that:

- *A necessary and sufficient condition for* $\lim_{t\to\infty}\mu_t$ *to exist in the weak* $(C_b(\mathbb{R}^3))^*$ *topology is that* μ_0 *has finite energy.*

This statement extends a result [12] of two of the authors from the Kac equation to the Boltzmann equation. See also [8] for a quantitative study for the Kac equation.

Notice that the estimate on the rate at which the mass explodes to infinity is governed by the rate that η_L diverges as L tends to infinity. If this divergence is very slow, so that the energy is "just barely infinite", then the explosion of the mass to infinity will be correspondingly slow. Indeed, whenever the energy is finite, there is convergence to a Maxwellian, and so there is no explosion in this case. However, as shown in [9], if the tails on the initial distribution are long, so that the energy is "just barely finite", then the convergence can be arbitrarily slow.

We have taken care to formulate Theorem 1.1 in complete generality for measure valued initial data. However, when proving theorems about measure valued solutions of (1.1), much of the work can often be done estimating solutions with quite smooth initial data. This is because the Wild convolution has an important "commutativity" property:

Let $M^{(\epsilon)}$ be the Maxwellian density

$$M^{(\epsilon)}(v) = \left(\frac{1}{2\pi\epsilon}\right)^{3/2} e^{-|v|^2/(2\epsilon)}.$$

Then one has the frequently useful *Bobylev's identity* [1], which says that for all probability measures μ and ν, with $*$ denoting the *standard* convolution of probability densities,

$$(\mu\circ\nu) * M^{(\epsilon)} = (\mu * M^{(\epsilon)})\circ(\nu * M^{(\epsilon)}).$$

It follows that for all n,

$$Q_n^+(\mu * M^{(\epsilon)}) = (Q_n^+(\mu)) * M^{(\epsilon)}.$$

Because of (1.6), this means that convolving the initial data with $M^{(\epsilon)}$ and then solving the equation, yields the same result as does first solving the equation, and then convolving with $M^{(\epsilon)}$. In other words, for any $t>0$, let $f^{(\epsilon)}(v,t)$ be the solution of (1.1) with initial data $\mu_0 * M^{(\epsilon)}$, and let μ_t be the solution of (1.1) with initial data μ_0. Then

$$\mu_t * M^{(\epsilon)} = f^{(\epsilon)}(\cdot,t).$$

Because of this identity, it will be easy to prove Theorem 1.1 if we can first prove it for initial data of the form $\mu_0 * M^{(\epsilon)}$, for some $\epsilon>0$ – even $\epsilon = 1$ will do, as we shall see.

The Wild summation formula reduces many questions about the continuous time evolution

$$t \mapsto f(\cdot,t) \tag{1.12}$$

described by (1.5) to questions about the sequence

$$n \mapsto Q_n^+(f_0). \tag{1.13}$$

The strategy of rephrasing questions about (1.12) as questions about (1.13) was championed by McKean, [13, 14] who proved a very useful alternate formula for $Q_n^+(f_0)$. McKean's formula is not recursive, but instead involves a certain random walk on graphs:

$$Q_n^+(f_0) = \sum_{\gamma\in\Gamma_n} \mathrm{Pr}_{\Gamma_n}(\gamma) C_\gamma(f_0, \dots, f_0), \tag{1.14}$$

where Pr_{Γ_n} is the probability that a certain random walk on graphs passes through γ, and $C_\gamma(f_0, \dots, f_0)$ is an n-fold Wild convolution of f_0 determined by γ. The reason for bringing in the graphs γ is that the Wild convolution is non–associative, so that in general

$$(f \circ f) \circ (f \circ f) \neq ((f \circ f) \circ f) \circ f.$$

McKean's graphs index the different ways of associating the pairs in an n-fold Wild convolution. See [14], and also [4, 5], which further develop McKean's approach, for additional explanation, and pictures of the graphs. The n-fold Wild convolutions $C_\gamma(f, \dots, f)$ are linear in each of the arguments. We shall make use of this later on, which is why we have written out all of the factors in the argument of C_γ.

The sum in (1.14) provides a basis for separating $Q_n^+(f)$ into two pieces: one "good" and one "bad", in which the good part has some nice property, such as being smooth, or being close to a Maxwellian, and the bad one has a small total mass. This was the strategy employed in [5]. Our main result here is of this type, but we must use a further stochastic decomposition of $Q_n^+(f)$ that we introduce in the next section. Using this stochastic decomposition, we prove:

Theorem 1.2 *Let μ_0 be a probability measure on $\mathbb{R}^3$ such that $\int_{\mathbb{R}^3} |v|^2 \mathrm{d}\mu(v) = \infty$. Let η_L be defined by* (1.9). *For any fixed $\epsilon > 0$, let $f_0(v) = \mu_0 * M^{(\epsilon)}$. Then there is an n_0 depending only on ϵ such that for all $n \geq n_0$, there is a decomposition of $Q_n^+(f_0)$ of the form*

$$Q_n^+(f_0) = q_n G_n(f_0) + (1 - q_n) B_n(f_0) \tag{1.15}$$

where $G_n(f_0)$ and $B_n(f_0)$ are probability densities, and moreover, for universal, finite constants C_1 and C_2, it is the case that for any b with $0 < b < 1/3$ and with L_n defined by $L_n = n^{\kappa(1-3b)/5}$,

$$\|G_n(f_0)\|_\infty \leq \frac{C_1}{\eta_{L_n}^{3/2}} \quad \text{and} \quad q_n \leq \frac{C_2}{n^{\kappa b}}.$$

Once we have proved Theorem 1.2, Theorem 1.1 will easily follow, as we show in Sect. 7, where both theorems are proved. The next few sections establish lemmas and notation used in the proofs, and explain our strategy.

We conclude this section with a useful remark pointed out by a referee: The condition $\int_{\mathbb{R}^3} |v|^2 \mathrm{d}\mu(v) = \infty$ is consistent with $\int_{\mathbb{R}^3} |v|^s \mathrm{d}\mu(v) < \infty$ for $s < 2$. If furthermore $1 < s < 2$, then the momentum is well defined and finite, and is conserved by the solution, since $\int_{\mathbb{R}^3} |v|^s \mathrm{d}\mu_t(v) < \infty$ will at worst grow exponentially with t.

2 The Fourier Transform of $C_\gamma(f, \dots, f)$

Computing the Wild convolution of the densities f and g means computing the integral on the right hand side in (1.3). This involves both an integral over the velocity w, and an integral over the unit vector σ. In an n-fold Wild convolution $C_\gamma(f_1, \dots, f_n)$, there will be an integration over $n-1$ velocities, and an integration over $n-1$ unit vectors, $\sigma_1, \dots, \sigma_{n-1}$. In what follows, we need to separate these two integrals, doing first all of the velocity integrations with all of the unit vectors held fixed. The simplest way to do this is to use the Fourier transform.

There is an explicit formula, due to Bobylev [1], for the Fourier transform of the Wild convolution of two probability densities f and g:

$$\widehat{g \circ h}(\xi) = \int_{S^2} \widehat{g}(\xi_+)\widehat{h}(\xi_-)B(\sigma \cdot \xi/|\xi|)\mathrm{d}\sigma \tag{2.1}$$

where

$$\xi_+ = \frac{\xi + |\xi|\sigma}{2} \quad \text{and} \quad \xi_- = \frac{\xi - |\xi|\sigma}{2}.$$

One easily sees that with θ denoting the angle between ξ and σ,

$$|\xi_+|^2 = |\xi|^2 \cos^2(\theta/2) \quad \text{and} \quad |\xi_-|^2 = |\xi|^2 \sin^2(\theta/2). \tag{2.2}$$

We now wish to iterate the application of this formula to generate a formula for the Fourier transform of $C_\gamma(f)$. The only difficulty is notational. To be clear, consider first the simple McKean graph with three leaves such that the two leftmost leave have depth 2, and the leaf on the right has depth 1. In this case, $C_\gamma(f, g, h) = (f \circ g) \circ h$, which is all one really need to know about the McKean graphs at present. One easily iterates (2.1) and finds $\widehat{C_\gamma}(f, g, h)$ is obtained by integrating

$$\widehat{f}\left(\frac{\xi + |\xi|\sigma_2}{4} + \frac{|\xi + |\xi|\sigma_2|}{4}\sigma_1\right)\widehat{g}\left(\frac{\xi + |\xi|\sigma_2}{4} - \frac{|\xi + |\xi|\sigma_2|}{4}\sigma_1\right)\widehat{h}\left(\frac{\xi - |\xi|\sigma_2}{4}\right)$$

over $S^2 \times S^2$ with respect to the probability measure

$$B(\sigma_1 \cdot \xi/|\xi|)B(\sigma_2 \cdot \xi/|\xi|)\mathrm{d}\sigma_1\mathrm{d}\sigma_2.$$

We introduce the following notation for the arguments of $\widehat{f}$, $\widehat{g}$ and $\widehat{h}$: With γ denoting the McKean graph at hand, define

$$\begin{aligned} P_1(\xi, \gamma, \sigma_1, \sigma_2) &= \frac{\xi + |\xi|\sigma_2}{4} + \frac{|\xi + |\xi|\sigma_2|}{4}\sigma_1, \\ P_2(\xi, \gamma, \sigma_1, \sigma_2) &= \frac{\xi + |\xi|\sigma_2}{4} - \frac{|\xi + |\xi|\sigma_2|}{4}\sigma_1, \\ P_3(\xi, \gamma, \sigma_1, \sigma_2) &= \frac{\xi - |\xi|\sigma_2}{4}. \end{aligned} \tag{2.3}$$

More generally, if γ is any McKean graph with n leaves, it represents a convolution of n probability densities, with the structure of the graph encoding the order in which the iterated convolutions are done. With n factors, there will be $n-1$ unit vectors $\sigma_1, \dots, \sigma_{n-1}$ to integrate over. We need a notation for the argument of the ℓth factor, and so we define $P_\ell(\xi, \gamma, \sigma_1, \dots, \sigma_{n-1})$ to be the argument of the function corresponding to the ℓth leaf, $1 \le \ell \le n$. In any particular case, an explicit formula, such as is given in (2.3) can be worked out.

Returning to (2.3), we compute the magnitudes of the $P_j(\xi,\gamma,\sigma_1,\sigma_2)$. By using (2.2), we see that for our three leaved McKean graph,

$$|P_1(\xi,\gamma,\sigma_1,\sigma_2)|^2 = \cos^2(\theta_1/2)\cos^2(\theta_2/2)|\xi|^2.$$

Indeed, one easily sees from (2.2) that in general, for any γ, any ℓ, and any choice of the unit vectors, $|P_\ell(\xi,\gamma,\sigma_1,\dots,\sigma_n)|$ will be a multiple of $|\xi|$. Define the numbers $\pi_\ell(\gamma,\sigma_1,\dots,\sigma_n)$ by

$$|P_\ell(\xi,\gamma,\sigma_1,\dots,\sigma_n)|^2 = \pi_\ell^2(\gamma.\sigma_1,\dots,\sigma_n)|\xi|^2.$$

Each $\pi_\ell^2(\gamma,\sigma_1,\dots,\sigma_n)$ will be a product of up to $n-1$ factors, each of the form $\sin^2(\theta_j/2)$ or $\cos^2(\theta_j/2)$. Indeed, for our three leaved example

$$\begin{aligned} \pi_1^2(\gamma,\sigma_1,\sigma_2) &= \cos^2(\theta_1/2)\cos^2(\theta_2/2),\\ \pi_2^2(\gamma,\sigma_1,\sigma_2) &= \sin^2(\theta_1/2)\cos^2(\theta_2/2),\\ \pi_3^2(\gamma,\sigma_1,\sigma_2) &= \sin^2(\theta_2/2). \end{aligned} \tag{2.4}$$

Notice that

$$\pi_1^2(\gamma,\sigma_1,\sigma_2)+\pi_2^2(\gamma,\sigma_1,\sigma_2)+\pi_3^2(\gamma,\sigma_1,\sigma_2)=1,$$

uniformly in σ_1 and σ_2. It is easy to see that the analogous result holds for any McKean graph γ:

$$\sum_{\ell=1}^{n}\pi_\ell^2(\gamma,\sigma_1,\dots,\sigma_{n-1}) = 1. \tag{2.5}$$

We are almost ready to give a formula for $\widehat{C_\gamma}(f_1,\dots,f_n)$. We first define the probability measure $\mathrm{d}\beta_n$ on $[S^2]^{n-1}$ given by

$$\mathrm{d}\beta_n = \left(\prod_{\ell=1}^{n-1} B(\sigma_\ell\cdot\xi/|\xi|)\right)\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_{n-1}. \tag{2.6}$$

Then with $\vec{\sigma}$ denoting the $(n-1)$-tuple $(\sigma_1,\dots,\sigma_{n-1})$ in $[S^2]^{n-1}$, we have the formula

$$\widehat{C_\gamma}(f_1,\dots,f_n)(\xi) = \int_{[S^2]^{n-1}}\prod_{\ell=1}^{n}\widehat{f_\ell}(P_\ell(\xi,\gamma,\vec{\sigma}))\mathrm{d}\beta_n. \tag{2.7}$$

Note that for each fixed $\vec{\sigma}$, $\prod_{\ell=1}^n \widehat{f_\ell}(P_\ell(\xi,\gamma,\vec{\sigma}))$ is the Fourier transform of a probability density. This is the density one gets if one just integrates over the velocities in forming the Wild convolutions, and holds the unit vectors $\vec{\sigma}$ fixed. Let us define this density to be $C_{\gamma,\vec{\sigma}}(f_1,\dots,f_n)$ so that

$$\widehat{C_{\gamma,\vec{\sigma}}}(f_1,\dots,f_n)(\xi) = \prod_{\ell=1}^{n}\widehat{f_\ell}(P_\ell(\xi,\gamma,\vec{\sigma})). \tag{2.8}$$

Clearly, $C_{\gamma,\vec{\sigma}}(f_1,\dots,f_n)$ and $C_\gamma(f_1,\dots,f_n)$ are related by

$$C_\gamma(f_1,\dots,f_n) = \int_{[S^2]^{n-1}}[C_{\gamma,\vec{\sigma}}(f_1,\dots,f_n)]\mathrm{d}\beta_n. \tag{2.9}$$

In addition to the identity (2.5), the other important thing to know concerning the $\pi_\ell(\gamma,\vec{\sigma})$ is that none of them are too large: For "most" γ and $\vec{\sigma}$, most of the $\pi_\ell(\gamma,\vec{\sigma})$ are of *roughly* same size. We will prove estimates that make this precise in Sect. 4. In the mean time, an optimist might hope that *in some sense, for at least most* $\vec{\sigma}\in[S^2]^{n-1}$ *and most* $\gamma\in\Gamma_n$, *and most* ℓ,

$$|\pi_\ell(\gamma,\sigma)|\approx\frac{1}{n^{1/2}}. \tag{2.10}$$

To the extent that this is true, we would have

$$\widehat{C_\gamma(f)}(\xi)\approx(\widehat{f}(\xi/n^{1/2}))^n.$$

Then, in case f were a mean zero probability density of finite variance σ, a classical argument would give

$$\lim_{n\to\infty}(\widehat{f}(\xi/n^{1/2}))^n=e^{-2\pi^2\sigma^2|\xi|^2}.$$

This, together with (1.14), would suggest that, to the extent that (2.10) is true for most γ (with respect to Pr_{Γ_n}), and all sufficiently large n,

$$Q_n^+(f)(v)\approx\left(\frac{1}{2\pi\sigma^2}\right)^{3/2}e^{-|v|^2/2\sigma^2}. \tag{2.11}$$

Such results, suggested by McKean [13, 14], have been proved in the case of finite variance, which in physical terms means finite energy; see [4, 5, 9] and [12]. Here we take up the case of infinite energy. In the case that σ is infinite, (2.11) suggests that one should expect

$$\lim_{n\to\infty}Q_n^+(f)=0,$$

"nearly uniformly". In the next sections, we shall produce a precise rendering of this intuitive picture, and thus prove Theorem 1.2.

3 The Stochastic Decomposition

Consider a probability density f on $\mathbb{R}^3$ such that $\int_{\mathbb{R}^3}|v|^2f(v)\mathrm{d}v=\infty$. For each $L>0$, define

$$p_L=\int_{|v|\le L}f(v)\mathrm{d}v. \tag{3.1}$$

We shall be interested in large values of L for which p_L is close to 1. In any case, for all L large enough that $p_L>0$, we can define

$$g_{(L)}(v)=\frac{1}{p_L}1_{|v|\le L}f(v), \tag{3.2}$$

$$\eta_L=\int_{|v|\le L}|v|^2f(v)\mathrm{d}v, \tag{3.3}$$

and

$$\sigma_L^2=\inf_{w\in\mathbb{R}^3}\int_{\mathbb{R}^3}|v-w|^2g_{(L)}(v)\mathrm{d}v. \tag{3.4}$$

Our first Lemma assures us that for large L, σ_L^2 is not too much smaller than η_L.

Lemma 3.1 *For all* $\epsilon > 0$, *there is an* $L_1 < \infty$ *such that for all* $L > L_1$,

$$\sigma_L^2 \geq \frac{1-\epsilon}{2}\eta_L. \tag{3.5}$$

Proof First, for any $c < 1$, choose L_0 so that $p_{L_0} \geq c$. Then for any b with $0 < b < 1$, choose L_1 so that

$$b^2\eta_{L_1} > 4L_0^2, \tag{3.6}$$

which is of course possible since η_L diverges to infinity as L increases.

The infimum in (3.4) is achieved at $w = \mu_L$ where

$$\mu_L = \int_{\mathbb{R}^3} v g_{(L)}(v)\mathrm{d}v. \tag{3.7}$$

For any $L \geq L_1$, we consider two cases.

First, suppose that $|\mu_L| \geq (\eta_L/2)^{1/2}$. Then,

$$\begin{aligned}
\sigma_L^2 &= \int_{\mathbb{R}^3} |\mu_L - v|^2 g_L(v)\mathrm{d}v \\
&\geq \int_{|v|\leq L_0} |\mu_L - v|^2 g_L(v)\mathrm{d}v \\
&\geq ||\mu_L| - L_0|^2 \frac{p_{L_0}}{p_L} \\
&\geq c|\mu_L|(|\mu_L| - 2L_0) \\
&\geq c(\eta_L/2)^{1/2}((\eta_L/2)^{1/2} - b(\eta_L)^{1/2}).
\end{aligned} \tag{3.8}$$

For c sufficiently close to 1 and b sufficiently close to zero, the right hand side exceeds $\frac{1-\epsilon}{2}\eta_L$.

It remains to consider the case $|\mu_L| < (\eta_L/2)^{1/2}$. But then the identity

$$\sigma_L^2 = \int_{\mathbb{R}^3} (v^2 - \mu_L^2) f_{(L)}(v)\mathrm{d}v = \frac{1}{p_L}\eta_L - \mu_L^2,$$

yields $\sigma_L^2 \geq (1/p_L - 1/2)\eta_L > \eta_L/2$. □

Next, with μ_L given by (3.7), define

$$f_{(L)}(v) = \frac{1}{p_L} 1_{\{|v+\mu_L|\leq L\}} f(v+\mu_L), \tag{3.9}$$

$$f^{(L)}(v) = \frac{1}{1-p_L} 1_{\{|v+\mu_L|\geq L\}} f(v+\mu_L). \tag{3.10}$$

With these definitions, Lemma 3.1, applied with $\epsilon = 1/2$, assures us that

$$\int_{\mathbb{R}^3} v f_{(L)}(v)\mathrm{d}v = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} |v|^2 f_{(L)}(v)\mathrm{d}v \geq \frac{1}{4}\eta_L.$$

Let $\{\alpha_j\}_{j\in\mathbb{N}}$ be an independent identically distributed sequence of Bernoulli random variables with success probability

$$\Pr(\alpha_j = 1) = p_L.$$

Then, for each L, j and v,

$$\mathrm{E}_{\mathrm{Ber}}[\alpha_j f_{(L)}(v) + (1-\alpha_j) f^{(L)}(v)] = f(v+\mu_L),$$

where $\mathrm{E}_{\mathrm{Ber}}$ denotes the expectation with respect to the law of our Bernoulli variables.

Now since the L^∞ and L^1 bounds to be proved in Theorem 1.2 are translation invariant, and since the variance is invariant under translation, we may assume, without loss of generality, that $\mu_L = 0$. *This will simplify the following computations, and we henceforth work under this assumption.*

By the independence of the Bernoulli variables, for any $\gamma \in \Gamma_n$,

$$C_\gamma(f,\dots,f) = \mathrm{E}_{\mathrm{Ber}}[C_\gamma(\alpha_1 f_{(L)} + (1-\alpha_1)f^{(L)},\dots,\alpha_n f_{(L)} + (1-\alpha_n)f^{(L)})]. \tag{3.11}$$

We are finally in a position to introduce our stochastic decomposition, for which we first introduce an appropriate probability space. Let $(\Omega_{\mathrm{Ber}}, \Pr_{\mathrm{Ber}})$ denote the probability space of our Bernoulli process. For each n, define

$$\Omega_n = \Omega_{\mathrm{Ber}} \times \Gamma_n \times [S^2]^{n-1},$$

and on Ω_n define the probability measure

$$\Pr_n = \Pr_{\mathrm{Ber}} \otimes \Pr_{\Gamma_n} \otimes \mathrm{d}\beta_n.$$

Let E_n denote the expectation with respect to $\Pr_n$.

Because of the way that $\Pr_n$ incorporates $\Pr_{\Gamma_n}$,

$$Q_n^+(f) = \mathrm{E}_n[C_\gamma(f,\dots,f)].$$

Because of the way that $\Pr_n$ incorporates $\mathrm{d}\beta_n$, we have from (2.9) that

$$Q_n^+(f) = \mathrm{E}_n[C_{\gamma,\vec{\sigma}}(f,\dots,f)].$$

Finally, because of the way that $\Pr_n$ incorporates $\Pr_{\mathrm{Ber}}$, we have from (3.11)

$$Q_n^+(f) = \mathrm{E}_n[C_{\gamma,\vec{\sigma}}(\alpha_1 f_{(L)} + (1-\alpha_1)f^{(L)},\dots,\alpha_n f_{(L)} + (1-\alpha_n)f^{(L)})]. \tag{3.12}$$

Now, for any event $\Lambda \subset \Omega_n$, define $Q_n^+(f,\Lambda)$ by conditioning on Λ:

$$Q_n^+(f,\Lambda) = \mathrm{E}_n[(C_{\gamma,\vec{\sigma}}(\alpha_1 f_{(L)} + (1-\alpha_1)f^{(L)},\dots,\alpha_n f_{(L)} + (1-\alpha_n)f^{(L)}))|\Lambda]. \tag{3.13}$$

Evidently,

$$Q_n^+(f) = \Pr_n(\Lambda)Q_n^+(f,\Lambda) + \Pr_n(\Lambda^c)Q_n^+(f,\Lambda^c). \tag{3.14}$$

Compare this with the decomposition of $Q_n^+(f)$ in (1.15). To prove Theorem 1.2, it suffices to show that for all sufficiently large n, we can choose an event $\Lambda \subset \Omega_n$ so that

$$\Pr_n(\Lambda^c) \quad \text{and} \quad \|Q_n^+(f,\Lambda)\|_\infty$$

are both very small.

To show that $\|Q_n^+(f,\Lambda)\|_\infty$ is very small, we show that $\widehat{Q_n^+}(f,\Lambda)$ is integrable, and that in fact, $\|\widehat{Q_n^+}(f,\Lambda)\|_1$ is very small. Then the Fourier inversion theorem implies that

$$\|Q_n^+(f,\Lambda)\|_\infty \le \|\widehat{Q_n^+}(f,\Lambda)\|_1.$$

Now, by (3.13) and linearity,

$$\widehat{Q_n^+}(f,\Lambda) = \mathrm{E}_n[(\widehat{C_{\gamma,\vec\sigma}}(\alpha_1 f_{(L)} + (1-\alpha_1)f^{(L)},\dots,\alpha_n f_{(L)} + (1-\alpha_n)f^{(L)})|\Lambda]. \tag{3.15}$$

Next, notice that the Fourier transform of $[\alpha_j f_{(L)}(v) + (1-\alpha_j)f^{(L)}(v)]$ is simply given by

$$\widehat{f_{(L)}}(\alpha_j\xi)\widehat{f^{(L)}}((1-\alpha_j)\xi),$$

since if $\alpha_j = 1$, the second factor is 1, while if $\alpha_j = 0$, the first factor is 1. Therefore, from (2.8),

$$\begin{aligned}&\widehat{C_{\gamma,\vec\sigma}}(\alpha_1 f_{(L)} + (1-\alpha_1)f^{(L)},\dots,\alpha_n f_{(L)} + (1-\alpha_n)f^{(L)}))\\ &\quad = \prod_{\ell=1}^n \widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi,\gamma,\vec\sigma))\widehat{f^{(L)}}((1-\alpha_\ell)P_\ell(\xi,\gamma,\vec\sigma)).\end{aligned}$$

Going back to (3.15), we have

$$\widehat{Q_n^+}(f,\Lambda)(\xi) = \mathrm{E}_n\left[\prod_{\ell=1}^n \widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi,\gamma,\vec\sigma))\widehat{f^{(L)}}((1-\alpha_\ell)P_\ell(\xi,\gamma,\vec\sigma))|\Lambda\right] \tag{3.16}$$

Now let M_L be the centered Maxwellian with variance σ_L:

$$M_L(v) = \left(\frac{1}{2\pi\sigma_L^2}\right)^{3/2} e^{-|v|^2/2\sigma_L^2}. \tag{3.17}$$

Define

$$\widehat{S_n}(f,\Lambda)(\xi) = \mathrm{E}_n\left[\prod_{\ell=1}^n \widehat{M_L}(\alpha_\ell P_\ell(\xi,\gamma,\vec\sigma))\widehat{f^{(L)}}((1-\alpha_\ell)P_\ell(\xi,\gamma,\vec\sigma))|\Lambda\right]. \tag{3.18}$$

In passing from (3.16) to (3.18), we have simply substituted each factor of $\widehat{f_{(L)}}$ in (3.16) with $\widehat{M_L}$ evaluated at the same argument.

Next, introduce the random functions

$$H_n(\xi) = \left|\prod_{\ell=1}^n \widehat{f_{(L)}}(\alpha_\ell P(\xi,\gamma,\vec\sigma)) - \prod_{\ell=1}^n \widehat{M_L}(\alpha_\ell P(\xi,\gamma,\vec\sigma))\right| \tag{3.19}$$

and

$$J_n(\xi) = \left|\prod_{\ell=1}^n \widehat{M_L}(\alpha_\ell P(\xi),\gamma,\vec\sigma)\right|. \tag{3.20}$$

- *To facilitate reading of the expressions that follow, we shall suppress the explicit reference to dependence on γ and $\vec\sigma$.*

Using the trivial bound $\|\widehat{f^{(L)}}\|_\infty = 1$, one sees that

$$|\widehat{Q_n^+}(f,\Lambda)(\xi) - \widehat{S}_n(f,\Lambda)(\xi)| \le \mathrm{E}_n[H_n(\xi)|\Lambda] \tag{3.21}$$

and

$$|\widehat{S}_n(f,\Lambda)(\xi)| \le \mathrm{E}_n[J_n(\xi)|\Lambda]. \tag{3.22}$$

It follows that

$$|\widehat{Q_n^+}(f,\Lambda)(\xi)| \le \mathrm{E}_n[H_n(\xi) + J_n(\xi)|\Lambda]. \tag{3.23}$$

We shall show that $\mathrm{E}_n[H_n(\xi)|\Lambda]$ is small because the substitution of $\widehat{f_{(L)}}$ by $\widehat{M_L}$ in passing from (3.16) to (3.18) has only a small effect, for the same reason that the corresponding replacement in the proof of the Central Limit Theorem has only a small effect.

Moreover,

$$\widehat{M_L}(\xi) = e^{-2\pi^2\sigma_L^2|\xi|^2},$$

which decays very rapidly. As long as there are plenty of such factors in the random product J_n, one can expect J_n to have a very small L^1 norm.

In the next section, we prove the probabilistic estimate that will yield us an appropriate choice of Λ for each n.

4 Probabilistic Lemmas

We begin with a lemma which assures us that, with high probability, none of the individual terms in the sum

$$\sum_{\ell=1}^{n} \pi_\ell^2 = 1$$

makes a very large contribution.

Lemma 4.1 *There is a constant $C < \infty$ such that for all n,*

$$\mathrm{E}_n\left[\sum_{\ell=1}^{n} (\pi_\ell(\gamma,\vec{\sigma}))^4\right] \le \frac{C}{n^\kappa}, \tag{4.1}$$

where

$$\kappa = 1 - \int_{S^2} [\cos^4(\theta/2) + \sin^4(\theta/2)] B(\cos(\theta))\mathrm{d}\sigma. \tag{4.2}$$

Remark As was shown in [6], the constant κ is the absolute value of the spectral gap for the linearized collision operator.

Proof For any McKean graph γ with n leaves, define the random variable W by $W = \sum_{\ell=1}^{n} (\pi_\ell(\gamma,\vec{\sigma}))^4$, where each $\pi_\ell(\gamma,\vec{\sigma})$ is a product of sines and cosines, as in (2.4).

Define $T(n) = \mathrm{E}_n(W)$. We shall show, using an argument from [4] that $T(n)$ solves the recurrence relation

$$T(n) = \frac{1}{n-1}\left(\frac{1-\kappa}{2}\sum_{j=1}^{n-1} T(j)T(n-j)\right), \tag{4.3}$$

where $T(1) = 1$.

To prove (4.3), we take any McKean graph γ, and "split" it into two graphs, γ_{left} and γ_{right} by removing the root node. (This is the exact same procedure that was used in [4] to estimate a very similar quantity, and [4] may be consulted for further details and pictures.) If γ is selected according to Pr_{Γ_n}, then as shown in [4], the number of leaves in γ_{left} (and hence in γ_{right}) is uniformly distributed in $\{1, \ldots, n-1\}$. Also, all of the terms π_ℓ contributing to $W(\gamma_{\mathrm{left}})$ and $W(\gamma_{\mathrm{right}})$ are the same as the corresponding terms contributing to $W(\gamma)$, except that they lack a factor of $\cos^2(\theta_1/2)$ on the left, and $\sin^2(\theta_1/2)$ on the right. Therefore,

$$W(\gamma) = (\cos(\theta_1/2))^4 W(\gamma_{\mathrm{left}}) + (\sin(\theta_1/2))^4 W(\gamma_{\mathrm{right}}).$$

Now, taking the expectation, symmetrizing, and using the uniform distribution of the number of leaves on the left (and hence right), we obtain (4.3).

The solution of this recurrence relation was estimated in [4], but in the meantime, an exact solution has been found in [11]. The result is

$$T(n) = \frac{\Gamma(n-\kappa)}{\Gamma(1-\kappa)\Gamma(n)}$$

where $\Gamma(\cdot)$ denotes the usual gamma function.

It follows that for all n sufficiently large,

$$T(n) \le \frac{2}{\Gamma(1-\kappa)}\frac{1}{n^\kappa},$$

from which the result easily follows. □

We next make some definitions that will be crucial in what follows. For any $a > 0$, define the random subset G_a of $\{1, \ldots, n\}$ by

$$G_a = \{\ell : \pi_\ell \le a/n^{\kappa/2}\}. \tag{4.4}$$

Then, for any $A > 0$ and any $a > 0$, define the event

$$\Lambda_{A,a} \subset \Omega_n$$

by

$$\Lambda_{A,a} = \left\{\sum_{\ell\in G_a} \alpha_\ell \pi_\ell^2 \ge \frac{p_L}{2} - \frac{A}{a^2}\right\} \cap \left\{\sum_{\ell=1}^{n} \pi_\ell^4 \le \frac{A}{n^\kappa}\right\}. \tag{4.5}$$

For an appropriate choice of A and a as powers of n, the event $\Lambda_{A,a}$ will be the "good" event on which we can successfully estimate H_n and J_n.

Lemma 4.2 *Let C be the constant in* (4.1). *Then*

$$\Pr_n(\Lambda_{A,a}) \geq 1 - 4C\frac{1-p_L}{p_L}\frac{1}{n^\kappa} - \frac{C}{A}. \tag{4.6}$$

Proof We first prove that

$$\Pr_n\left\{\sum_{\ell=1}^n \alpha_\ell \pi_\ell^2 \leq p_L/2\right\} \leq 4C\frac{1-p_L}{p_L}\frac{1}{n^\kappa}. \tag{4.7}$$

To see this, note that

$$\sum_{\ell=1}^n \alpha_\ell \pi_\ell^2 - p_L = \sum_{\ell=1}^n (\alpha_\ell - p_L)\pi_\ell^2$$

and so by Markov's inequality, the probability that $|\sum_{\ell=1}^n \alpha_\ell \pi_\ell^2 - p_L|$ exceeds $p_L/2$ is no greater than

$$\mathrm{E}_n\left(\frac{|\sum_{\ell=1}^n(\alpha_\ell - p_L)\pi_\ell^2|^2}{(p_L/2)^2}\right) = \mathrm{E}_n\left(\frac{p_L(1-p_L)\sum_{\ell=1}^n \pi_\ell^4}{(p_L/2)^2}\right) \leq 4C\frac{1-p_L}{p_L}\frac{1}{n^\kappa}.$$

We have used the fact that the $(\alpha_\ell - p_L)$ are independent, mean zero random variables to eliminate the cross terms in the expectation.

Next, once more by (4.1) and Markov's inequality,

$$\Pr_n\left(\sum_{\ell=1}^n \pi_\ell^4 \geq \frac{A}{n^\kappa}\right) \leq \frac{C}{A}. \tag{4.8}$$

Finally, to get a lower bound on $\sum_{\ell\in G_a} \alpha_\ell \pi_\ell^2$, we use (4.7) and an upper bound on $\sum_{\ell\in B_a} \alpha_\ell \pi_\ell^2$, where B_a denotes the complement of G_a in $\{1,\ldots,n\}$:

$$\sum_{\ell\in B_a} \alpha_\ell \pi_\ell^2 \leq \sum_{\ell\in B_a} \pi_\ell^4 \frac{n^\kappa}{a^2} \leq \frac{n^\kappa}{a^2}\sum_{\ell=1}^n \pi_\ell^4,$$

so that on the event $\{\sum_{\ell=1}^n \pi_\ell^4 \leq A/n^\kappa\}$,

$$\sum_{\ell\in B_a} \alpha_\ell \pi_\ell^2 \leq \frac{A}{a^2}. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9) in the obvious way, we have the result. □

Lemma 4.3 *Suppose n, L, A and a are such that*

$$\frac{p_L}{2} - \frac{A}{a^2} - \frac{a^2}{n^\kappa} \geq \frac{1}{4}. \tag{4.10}$$

Then for all outcomes in the event $\Lambda_{A,a}$ defined in (4.5), *there are* (*at least*) *two indices, ℓ_1 and ℓ_2, such that*

$$\alpha_\ell \pi_\ell^2 \geq \frac{1}{4n}$$

for $\ell = \ell_1, \ell_2$.

Proof By the definition of $\Lambda_{A,a}$, when (4.10) is true, $\sum_{\ell\in G_a}\alpha_\ell\pi_\ell^2 \geq 1/4$ for all outcomes in $\Lambda_{A,a}$ There are at most n terms in this sum, and so at least one is as large as $1/4n$.

Now delete this term from the sum. Since the term came from G_a, the loss is no greater than a^2/n^κ. Hence when (4.10) is true, the sum over the remaining terms is still at least $1/4$, and so there is a second term in B_a contributing at least $1/4n$. □

5 Inner Estimates

By a telescoping sum argument,

$$H_n(\xi) \leq \sum_{\ell=1}^{n}\left(\prod_{j=1}^{\ell-1}|\widehat{f_{(L)}}(\alpha_j P_j(\xi))|\right)|\widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi)) - \widehat{M_L}(\alpha_\ell P_\ell(\xi))|\left(\prod_{k=\ell+1}^{n}\widehat{M_L}(\alpha_k P_k(\xi))\right). \tag{5.1}$$

By convention, products over empty ranges of indices are defined to be 1.

We now seek upper bounds, for each ℓ, on

$$\left(\prod_{j=1}^{\ell-1}|\widehat{f_{(L)}}(\alpha_j P_j(\xi))|\right)\left(\prod_{k=\ell+1}^{n}\widehat{M_L}(\alpha_k P_k(\xi))\right) \tag{5.2}$$

and also on

$$|\widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi)) - \widehat{M_L}(\alpha_\ell P_\ell(\xi))|. \tag{5.3}$$

By a standard Taylor expansion argument,

$$|\widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi)) - \widehat{M_L}(\alpha_\ell P_\ell(\xi))| \leq K_L\frac{(2\pi)^3}{6}\alpha_\ell\pi_\ell^3|\xi|^3 \tag{5.4}$$

and

$$|\widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi)) - (1 - 2\pi^2\sigma_L^2\alpha_\ell\pi_\ell^2|\xi|^2)| \leq K_L\frac{(2\pi)^3}{6}\alpha_\ell\pi_\ell^3|\xi|^3 \tag{5.5}$$

where

$$K_L = \int_{\mathbb{R}^3}|v|^3(f_{(L)} + M_L)\mathrm{d}v.$$

Lemma 5.1 *For all $|\xi|$ with $|\xi| \leq \frac{\sigma_L^2}{2\pi a K_L}n^{\kappa/2}$, and all outcomes in $\Lambda_{A,a}$,*

$$\left(\prod_{j=1}^{\ell-1}|\widehat{f_{(L)}}(\alpha_j P_j(\xi))|\right)\left(\prod_{k=\ell+1}^{n}\widehat{M_L}(\alpha_k P_k(\xi))\right) \leq \exp\left(-\pi^2\sigma_L^2\left(\frac{p_L}{2} - \frac{A}{a^2} - \frac{a^2}{n^\kappa}\right)|\xi|^2\right).$$

Proof The bound on $|\xi|$ that is stated in the hypothesis has been chosen so that the right hand side in (5.4) will be less that one half the size of the quadratic term in the Taylor expansion for $\widehat{f}_{(L)}$, or $\widehat{M}_L$, which is the same. Indeed, for $j \in G_a$, so that $\pi_j \leq a/n^{\kappa/2}$,

$$|\xi| \leq \frac{\sigma_L^2}{2\pi a K_L}n^{\kappa/2} \quad \Rightarrow \quad K_L\frac{(2\pi)^3}{6}\alpha_j\pi_j^3|\xi|^3 < \alpha_j\pi^2\sigma_L^2\pi_j^2|\xi|^2.$$

It follows from this, (5.5), and the elementary estimate $1-x \le e^{-x}$, that for $j \in G_a$,

$$|\xi| \le \frac{\sigma_L^2}{2\pi a K_L} n^{\kappa/2} \quad \Rightarrow \quad \left|\widehat{f_{(L)}}(\alpha_j P_j(\xi))\right| \le 1 - \pi^2\sigma_L^2\alpha_j\pi_j^2|\xi|^2 \le e^{-\pi^2\sigma_L^2\alpha_j\pi_j^2|\xi|^2}.$$

For j that does not belong to G_a, we still have the trivial bound $|\widehat{f_{(L)}}(\alpha_j P_j(\xi))| \le 1$. Of course for all ℓ, and all ξ, we have

$$\widehat{M_L}((\alpha_\ell P_\ell(\xi)) \le e^{-\pi^2\sigma_L^2\alpha_\ell\pi_\ell^2|\xi|^2}.$$

Now, going back to (5.2), we can conclude that

$$\left(\prod_{j=1}^{\ell-1} |\widehat{f_{(L)}}(\alpha_j P_j(\xi))|\right)\left(\prod_{k=\ell+1}^{n} \widehat{M_L}(\alpha_k P_k(\xi))\right) \le \exp\left(-\pi^2\sigma_L^2\left(\sum_{j\in G_a}\alpha_j\pi_j^2 - a^2/n^\kappa\right)|\xi|^2\right).$$

The point of the subtracted term is that ℓ might belong to G_a. The result now follows from the definition of $\Lambda_{A,a}$ □

We now come to the main lemma of this section:

Lemma 5.2 *Suppose n, L, A and a are such that* (4.10) *holds. Then for all outcomes in $\Lambda_{A,a}$,*

$$\int_{|\xi|\le\sigma_L^2 n^{\kappa/2}/(2\pi a K_L)} H_n(\xi)\mathrm{d}\xi \le D\frac{K_L A^{1/2}}{\sigma_L^6 n^{\kappa/2}} \tag{5.6}$$

where

$$D = \left(K_L\frac{(2\pi)^3}{6}\right)\int_{\mathbb{R}^3}|\xi|^3\exp(-\pi^2|\xi|^2/4)\mathrm{d}\xi.$$

Proof Suppose that $|\xi| \le \sigma_L^2 n^{\kappa/2}/(2\pi a K_L)$. By (5.1), Lemma 5.1, and the hypothesis that (4.10) holds, we have for all outcomes in $\Lambda_{A,a}$,

$$H_n(\xi) \le \left(\sum_{\ell=1}^{n} |\widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi)) - \widehat{M_L}(\alpha_\ell P_\ell(\xi))|\right)\exp(-\pi^2\sigma_L^2|\xi|^2/4).$$

Next, by (5.4),

$$\sum_{\ell=1}^{n} |\widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi)) - \widehat{M_L}(\alpha_\ell P_\ell(\xi))| \le K_L\frac{(2\pi)^3}{6}|\xi|^3\sum_{\ell=1}^{n}\pi_\ell^3.$$

But by the Schwarz inequality,

$$\sum_{\ell=1}^{n}\pi_\ell^3 = \sum_{\ell=1}^{n}(\pi_\ell^2)(\pi_\ell) \le \left(\sum_{\ell=1}^{n}\pi_\ell^4\right)^{1/2}\left(\sum_{\ell=1}^{n}\pi_\ell^2\right)^{1/2} = \left(\sum_{\ell=1}^{n}\pi_\ell^4\right)^{1/2}.$$

On $\Lambda_{A,a}$, $\sum_{\ell=1}^n \pi_\ell^4 \le A/n^\kappa$, and so on the set under consideration,

$$H_n(\xi) \le \frac{\sqrt{A}}{n^{\kappa/2}}\left(K_L\frac{(2\pi)^3}{6}\right)|\xi|^3\exp(-\pi^2\sigma_L^2|\xi|^2/4).$$

Integrating the right hand side over all of $\mathbb{R}^3$ and scaling σ_L out of the integrand,we have the claimed result. $\square$

6 The Outer Estimates

In this section we estimate $\int_{|\xi|\geq\sigma_L^2 n^{\kappa/2}/(2\pi a K_L)} H_n(\xi)\mathrm{d}\xi$ and $\|J_n\|_1$.

Lemma 6.1 *Suppose that for some $\epsilon > 0$, $f = \mu_0 * M^{(\epsilon)}$. Suppose also n, L, A and a are such that*

$$\frac{p_L}{2} - \frac{A}{a^2} - 2\frac{a^2}{n} \geq \frac{1}{4}. \tag{6.1}$$

Then for all outcomes in $\Lambda_{A,a}$,

$$\int_{|\xi|\geq\sigma_L^2 n^{\kappa/2}/(2\pi a K_L)} \left|\prod_{\ell=1}^{n} \widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi))\right| \mathrm{d}\xi \leq \exp\left(-\frac{n^\kappa}{2}\frac{c_L\sigma_L^4}{(2\pi a K_L)^2}\right) 16\pi^4 \left(\frac{L^3}{3\epsilon}\right)(2n)^{3/2} \tag{6.2}$$

and

$$\|J_n\|_1 = \int_{\mathbb{R}^3} \left|\prod_{\ell=1}^{n} \widehat{M_L}(\alpha_\ell P_\ell(\xi))\right| \mathrm{d}\xi \leq \frac{1}{\pi^{3/2}\sigma_L^3}, \tag{6.3}$$

where

$$c_L = \frac{1}{3}\left(\frac{\epsilon^{3/2}}{48\sigma_L^3}\right). \tag{6.4}$$

Note that

$$\int_{|\xi|\geq\sigma_L^2 n^{\kappa/2}/(2\pi a K_L)} H_n(\xi)\mathrm{d}\xi \leq \int_{|\xi|\geq\sigma_L^2 n^{\kappa/2}/(2\pi a K_L)} \left|\prod_{\ell=1}^{n} \widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi))\right| \mathrm{d}\xi + \int_{\mathbb{R}^3} \left|\prod_{\ell=1}^{n} \widehat{M_L}(\alpha_\ell P_\ell(\xi))\right| \mathrm{d}\xi,$$

so that the Lemma provides both of the estimates that we seek.

Proof The hypothesis that $f = \mu_0 * M^{(\epsilon)}$ has two consequences that shall be used here. There first is that f is then bounded. Indeed, $\|f\|_\infty \leq (2\pi\epsilon)^{-3/2}$. It follows that for all L under consideration here, $\|f_{(L)}\|_\infty \leq \epsilon^{-3/2}$.

The bound on $\|f_{(L)}\|_\infty$ allows us to apply Lemma 9.1 in the Appendix to $f_{(L)}$, and to conclude that for all $\eta \leq 1$,

$$|\xi| \geq \eta \quad \Longrightarrow \quad |\widehat{f_{(L)}}(\xi)| \leq 1 - c_L\eta^2 \tag{6.5}$$

where c_L is given by (6.4).

Notice that

$$|\xi| \geq \sigma_L^2 n^{\kappa/2}/(2\pi a K_L) \quad \Rightarrow \quad |P_\ell(\xi)| \geq \pi_\ell\sigma_L^2 n^\kappa/(2\pi a K_L).$$

Also, for $\ell \in G_a$,

$$\alpha_\ell \pi_\ell \sigma_L^2 n^\kappa/(2\pi a K_L) \le \sigma_L^2/(2\pi K_L) < 1.$$

Therefore, in the ℓth factor we use (6.5) with η given by

$$\eta = \pi_\ell \sigma_L^2 n^{\kappa/2}/(2\pi a K_L).$$

It follows that for $\ell \in G_a$ and $|\xi| \ge n^{\kappa/2}/(2\pi a K_L)$,

$$|\widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi))| \le 1 - n^\kappa (c_L \sigma_L^4 \alpha_\ell \pi_\ell^2/(2\pi a K_L)^2).$$

For $\ell \notin G_a$, we have the trivial estimate $\|\widehat{f_{(L)}}\|_\infty \le 1$. Consequently,

$$\left|\prod_{\ell=1}^n \widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi))\right| \le \exp\left(-n^\kappa \left(\sum_{\ell \in G_a} \alpha_j \pi_\ell^2\right) \frac{c_L \sigma_L^4}{(2\pi a K_L)^2}\right).$$

□

Since we have a good lower bound on $\sum_{\ell \in G_a} \alpha_j \pi_\ell^2$ on $\Lambda_{A,a}$, this provides an excellent pointwise bound on $H_n(\xi)$ for $|\xi| \ge n^{\kappa/2}\sigma_L^2/(2\pi a K_L)$. However, this region has an infinite volume, and we must bring in something else to estimate the integral in question.

At this point we make our second use of the hypothesis that $f = \mu_0 * M^{(\epsilon)}$. This entails that

$$\widehat{f}(\xi) \le G_\epsilon(\xi) \quad \text{where } G_\epsilon(\xi) = e^{-\epsilon|\xi|^2}.$$

Let h_L denote the function

$$h_L = \frac{1}{p_L} 1_{\{|v| \le L\}}.$$

By the definition of $f_{(L)}$,

$$|\widehat{f_{(L)}}(\xi)| = |(\widehat{f} * \widehat{h_L})(\xi)| \le (G_\epsilon * |\widehat{h_L}|)(\xi). \tag{6.6}$$

where $*$ denotes convolution.

Since both G_ϵ and $\widehat{h_L}$ are radial functions, so is $G_\epsilon * |\widehat{h_L}|$. Moreover,

$$\|G_\epsilon * |\widehat{h_L}|\|_2 \le \|G_\epsilon\|_1 \|\widehat{h_L}\|_2 = 2\left(\frac{\pi}{\epsilon}\right)^{3/2} \left(\frac{4\pi L^3}{3}\right)^{1/2} = 4\pi^2 \left(\frac{L^3}{3\epsilon}\right)^{1/2} \tag{6.7}$$

We know from Lemma 4.3 that when n, A and a are such that (4.10) holds, there are at least two indices ℓ_1 and ℓ_2 in G_a such that $\alpha_\ell \pi_\ell^2 \ge 1/(4n)$ for $\ell = \ell_1, \ell_2$. Therefore,

$$|\widehat{f_{(L)}}(\alpha_{\ell_1} P_{\ell 1}(\xi))||\widehat{f_{(L)}}(\alpha_{\ell_2} P_{\ell 2}(\xi))| \le (G_\epsilon * |\widehat{h_L}|)^2(|\xi|/(2\sqrt{n})).$$

It follows from (6.7) that

$$\int_{\mathbb{R}^3} (G_\epsilon * |\widehat{h_L}|)^2(|\xi|/(2\sqrt{n}))\mathrm{d}\xi \le 16\pi^4 \left(\frac{L^3}{3\epsilon}\right)(4n)^{3/2}. \tag{6.8}$$

Now let G_a^* denote G_a with the indices ℓ_1 and ℓ_2 removed. By Lemma 5.1, we have that on $\Lambda_{A,a}$, for all n, A and a such that (6.1) holds,

$$\begin{aligned}\left|\prod_{\ell=1}^{n}\widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi))\right| &\le \exp\left(-n^\kappa\left(\sum_{\ell\in G_a^*}\alpha_j\pi_\ell^2\right)\frac{c_L\sigma_L^4}{(2\pi a K_L)^2}\right)(G_\epsilon * |\widehat{h_L}|)^2(|\xi|/(2\sqrt{n}))\\ &\le \exp\left(-n^\kappa\left(\frac{p_L}{2}-\frac{A}{a^2}-2\frac{a^2}{n}\right)\frac{c_L\sigma_L^4}{(2\pi a K_L)^2}\right)(G_\epsilon * |\widehat{h_L}|)^2(|\xi|/(2\sqrt{n}))\\ &\le \exp\left(-\frac{n^\kappa}{2}\frac{c_L\sigma_L^4}{(2\pi a K_L)^2}\right)(G_\epsilon * |\widehat{h_L}|)^2(|\xi|/(2\sqrt{n})).\end{aligned}$$

It now follows from (6.8) that

$$\begin{aligned}&\int_{|\xi|\ge n^{\kappa/2}\sigma_L^2/(2\pi a K_L)}\left|\prod_{\ell=1}^{n}\widehat{f_{(L)}}(\alpha_\ell P_\ell(\xi,\gamma,\vec{\sigma}))\right| \mathrm{d}\xi\\ &\quad\le \exp\left(-\frac{n^\kappa}{2}\frac{c_L\sigma_L^4}{(2\pi a K_L)^2}\right)16\pi^4\left(\frac{L^3}{3\epsilon}\right)(2n)^{3/2}.\end{aligned}$$

It is much easier to estimate the integral in (6.3) since $\widehat{M_L}$ is radial and rapidly decaying. In fact,

$$|\widehat{M_L}(\alpha_\ell P_\ell(\xi)| = e^{-2\pi^2\sigma_L^2\alpha_\ell\pi_\ell^2|\xi|^2}.$$

Therefore, for all ξ,

$$\prod_{\ell=1}^{n}\widehat{M_L}(\alpha_\ell P_\ell(\xi,\gamma,\vec{\sigma})) \le \exp\left(-2\pi^2\left(\sum_{\ell\in G_a}\alpha_j\pi_\ell^2\right)\right).$$

It follows that for all outcomes in $\Lambda_{A,a}$, and all n, A and a such that (6.1) holds,

$$\prod_{\ell=1}^{n}\widehat{M_L}(\alpha_\ell P_\ell(\xi,\gamma,\vec{\sigma})) \le e^{-\pi^2\sigma_L^2|\xi|^2}.$$

7 Proofs of the Theorems

Proof of Theorem 1.2 We shall now obtain the decomposition (1.15) from (3.14) with

$$G_n(f_0) = Q_n^+(f_0,\Lambda_n),\qquad B_n(f_0) = Q_n^+(f_0,\Lambda_n^c),\qquad q_n = \mathrm{Pr}_n(\Lambda_n^c).$$

By Lemma 5.2, Lemma 6.1, and the remark following Lemma 6.1, for all n, L, A and a such that (6.1) holds,

$$\|\mathrm{E}[H_n|\Lambda_{A,a}]\|_1 \le \frac{D\sqrt{A}}{\sigma_L^6 n^{\kappa/2}} + \exp\left(-\frac{n^\kappa}{2}\frac{c_L\sigma_L^4}{(2\pi a K_L)^2}\right)16\pi^4\left(\frac{L^3}{3\epsilon}\right) + \frac{1}{\pi^{3/2}\sigma_L^3} \tag{7.1}$$

and

$$\|\mathrm{E}[J_n|\Lambda_{A,a}]\|_1 \le \frac{1}{\pi^{3/2}\sigma_L^3}. \tag{7.2}$$

We now choose L, A and a to grow like powers of n so that the right hand side of (7.1) tends to zero as n increases. To make such a choice, we first focus on the second term on the right, involving the exponential. We will make our choices of L_n, A_n and a_n so that

$$\frac{n^\kappa}{2}\frac{c_{L_n}\sigma_{L_n}^4}{(2\pi a_n K_{L_n})^2} \ge \mathrm{Const}{\cdot}n^b \tag{7.3}$$

for some $b > 0$. Then the middle term tends to zero faster than any power of n, and we need only concern ourselves with the remaining two terms. The last one will turn out to be dominant.

To see how to achieve (7.3), note that

$$K_L \le 2L\sigma_L^2 \quad \text{and} \quad \sigma_L \le L,$$

and also that c_L is proportional to σ_L^{-3} Thus,

$$\frac{c_L\sigma_L^4}{K_L^2} \ge \mathrm{Const}{\cdot}L^{-2}\eta_L^{-3} \ge \mathrm{Const}{\cdot}L^{-5}.$$

Given specific information on the divergence of η_L, we could make use of the middle bound, and reduce the factor of 5 in the definition of L_n in (7.4) below. However, for the sake of simplicity and generality, we use the bound on the right. Therefore we shall choose L_n and a_n so that $n^\kappa/(a_n^2L_n^5)$ is a positive power of n. We also need A_n/a_n^2 and a_n/n to tend to zero so that (6.1) will be satisfied for all sufficiently large n. We can achieve all of this by choosing, for any b with $0 < b < 1/3$,

$$L = L_n := n^{\kappa(1-3b)/5}, \qquad A = A_n := n^{\kappa b} \quad \text{and} \quad a = a_n := n^{\kappa b}. \tag{7.4}$$

We do this for all n sufficiently large that $n^{\kappa(1-3b)/5} \ge L_1$ as given in Lemma 3.1 For smaller values of n, we simply set $L_n = L_1$.

With these choices, for all large n, (7.3) holds, and (6.1) becomes

$$\frac{p_{L_n}}{2} - \frac{1}{n^{\kappa b}} - 2\frac{1}{n^{1-2\kappa b}} \ge \frac{1}{4}. \tag{7.5}$$

Since $\lim_{n\to\infty} p_{L_n} = 1$, it is clear that (6.1) will be satisfied for all sufficiently large n.

It follows that

$$\lim_{n\to\infty}\exp\left(-\frac{n^\kappa}{2}\frac{c_{L_n}\sigma_{L_n}^4}{(2\pi a_n K_{L_n})^2}\right)16\pi^4\left(\frac{L_n^3}{3\epsilon}\right) = 0,$$

and that the convergence takes place faster than any power of n.

The dominant term in (7.1) then is the last one, and so there is an n_0 so that for all $n \ge n_0$,

$$\|\mathrm{E}[H_n|\Lambda_{A_n,a_n}]\|_1 \le \frac{2}{\pi^{3/2}\sigma_L^3} \tag{7.6}$$

We now define

$$\Lambda_n = \Lambda_{A_n, a_n}.$$

Then, by Lemma 4.2,

$$\mathrm{Pr}_n(\Lambda_n^c) \leq \mathrm{Const}\cdot\frac{1}{n^{\kappa b}},$$

which gives us the desired bound on q_n. Moreover, from (3.23) and the estimates just above,

$$\|\widehat{Q_n^+}(f, \Lambda_n)\|_1 \leq \frac{3}{\pi^{3/2}\sigma_L^3} \tag{7.7}$$

for all $n \geq n_0$. By the Fourier inversion theorem, this entails,

$$\|Q_n^+(f, \Lambda_n)\|_\infty \leq \frac{3}{\pi^{3/2}\sigma_{L_n}^3} \tag{7.8}$$

for all $n \geq n_0$. By Lemma 3.1, this gives us the desired bound on $\|G_n(f_0)\|_\infty$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.1 We apply Theorem 1.2 with $\epsilon = 1$. We first consider $f(v,t) = M * \mu_t$. Let B_R denote the centered ball of radius R. We have that

$$\int_{B_R} Q_n^+(v)\mathrm{d}v \leq C_1\frac{R^3}{\eta_{L_n}^{3/2}} + C_2\frac{1}{n^{\kappa b}}$$

where b is any number with $0 < b < 1/3$, and $L_n = n^{\kappa(1-3b)/5}$. (We have absorbed a factor of $4\pi/3$ into the constant C_1 of Theorem 1.2 for convenience.)

Therefore,

$$\int_{B_R} f(v,t)\mathrm{d}v \leq C_1\left[e^{-t}\sum_{n=1}^{\infty}(1-e^{-t})^{n-1}\frac{R^3}{\eta_{L_n}^{3/2}}\right] + C_2\left[e^{-t}\sum_{n=1}^{\infty}(1-e^{-t})^{n-1}\frac{1}{n^{\kappa b}}\right].$$

The last term on the right is easy to estimate since for any $c > 0$,

$$e^{-t}\sum_{n=1}^{\infty} n^{-c}(1-e^{-t})^{n-1} \leq \mathrm{Const}\cdot e^{-ct}.$$

Applying this with $c = \kappa b$, we get a bound by a multiple of $e^{-\kappa bt}$.

If it is the case that η_L diverges like some power of L, necessarily less than quadratically, we can apply the same type of estimate, and can then optimize the choice of b. For instance if $\eta_L \geq \mathrm{Const}\cdot L^s$, we would get a bound by a multiple of $e^{-[3\kappa(1-3b)s/10]t}$. Then choosing $b = 3s/(10+9s)$, so that both exponents are the same, we would have decay bounded by $e^{-[3\kappa s/(10+9s)]t}$.

In general, we cannot be quite as explicit. But in any case, by the monotonicity of η_{L_n}, for any positive integer N,

$$e^{-t}\sum_{n=1}^{\infty}(1-e^{-t})^{n-1}\frac{R^3}{\eta_{L_n}^{3/2}} \leq Ne^{-t}\frac{R^3}{\eta_{L_1}^{3/2}} + \frac{R^3}{\eta_{L_N}^{3/2}},$$

which we get by breaking the sum at the Nth term. Thus,

$$e^{-t}\sum_{n=1}^{\infty}(1-e^{-t})^{n-1}\frac{R^3}{\eta_{L_n}^{3/2}} \le \inf_N\left(Ne^{-t}\frac{R^3}{\eta_{L_1}^{3/2}}+\frac{R^3}{\eta_{L_N}^{3/2}}\right),$$

which clearly tends to zero as t increases, but may do so very slowly if η_L diverges slowly with increasing L.

Thus, in any case, for every R,

$$\lim_{t\to\infty}\int_{B_R} f(v,t)\mathrm{d}v = 0.$$

To draw a conclusion for μ_t from this, we note that the convolution of the indicator function of B_R, 1_{B_R}, with M satisfies

$$1_{B_{2R}} * M \ge \phi(R)1_{B_R},$$

where $\phi(R)$ increases to 1 very rapidly. Thus,

$$\int_{B_{2R}} f(v,t)\mathrm{d}v = \int 1_{B_{2R}} * M\mathrm{d}\mu_t \ge \phi(R)\int_{B_R}\mu_t.$$

Therefore, we also have that

$$\lim_{t\to\infty}\int_{B_R}\mathrm{d}\mu_t = 0. \qquad \square$$

8 Remark on Eternal Solutions

In [2, 3], Bobylev and Cercignani constructed a family of self similar eternal solutions of the spatially homogeneous Boltzmann equation for Maxwellian molecules. Their solutions have the form

$$f(v,t) = e^{-3at}\Phi_a(e^{-at}v)$$

for certain numbers a. All of their solutions are infinite energy. As they remark, it is likely that their Φ_a is a probability density, as is suggested by our notation (and theirs) though their arguments only shows that it is a probability measure.

Our aim here is to establish *a–priori* bounds relating a and the tails of Φ_a. Fix some a, and in the rest of this section, let

$$\eta_L = \int_{|v|<L}\mathrm{d}\Phi_a(v).$$

Suppose that for some s, we have a bound

$$\eta_L \ge \mathrm{Const}{\cdot}L^s.$$

Then out main result gives us a bound on

$$\lim_{t\to\infty}\int_{|v|<R} e^{-3at}\mathrm{d}\Phi_a(e^{-at}v)$$

by a constant multiple of $R^3 e^{-[3\kappa s/(10+9s)]t}$. In particular,

$$\lim_{t\to\infty}\int_{|v|<e^{rt}} e^{-3at}\mathrm{d}\Phi_a(e^{-at}v) = 0$$

for any $r < \kappa s/(10+9s)$. Hence, for all sufficiently large t,

$$\int_{|v|<e^{(r-a)t}} \mathrm{d}\Phi_a(v) \le \frac{1}{2}.$$

In particular, $a > r$, an so, choosing r as large as possible,

$$a \ge \frac{\kappa s}{10+9s}.$$

For small values of a, the bound provides a limit on how long the tails of Φ_α can be.

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Appendix

This appendix contains a quantitative estimate on the decay of bounded probability densities of finite variance. It is based on a similar lemma in [7]. As elsewhere in the paper, we use the following convention for the Fourier transform in $\mathbb{R}^k$:

$$\hat{f}(\xi) = \int_{\mathbb{R}^k} e^{-2i\pi x\cdot\xi} f(x)\,dx.$$

Lemma 9.1 *Let g be a probability density on $\mathbb{R}^k$, such that*

$$\int_{\mathbb{R}^k} xg(x)\,\mathrm{d}x = \mu, \quad \text{and} \quad \int_{\mathbb{R}^k} |x-\mu|^2 g(x)\,\mathrm{d}x = \sigma^2.$$

Finally, suppose that $\|g\|_\infty < \infty$. Then, given η with $1 \ge \eta > 0$, there is a constant $\alpha(\|g\|_\infty, \sigma) > 0$ depending on g only through $\|g\|_\infty$ and σ, so that

$$|\xi| \ge \eta \quad \Longrightarrow \quad |\hat{g}(\xi)| \le 1 - \alpha(\|g\|_\infty, \sigma)\eta^2. \tag{9.1}$$

In the particular, one has the following explicit value for $\alpha(K,\sigma)$ in dimension $k=3$:

$$\alpha(\|g\|_\infty, \sigma) = \frac{1}{3}\left(\frac{1}{48\|g\|_\infty \sigma^3}\right) \quad \text{for } k=3. \tag{9.2}$$

Proof Let ξ be such that $|\xi| \ge \eta$, and let z be such that $\hat{g}(\xi)e^{-2i\pi z\cdot\xi} = |\hat{g}(\xi)|$. (Though z depends on ξ, which is fixed, we do not indicate this in our notation.) Then, with $\Re$ standing for real part,

$$
\begin{aligned}
|\hat{g}(\xi)| &= \Re[\hat{g}(\xi)e^{-2i\pi z\cdot\xi}] \\
&= \Re\left(\int_{\mathbb{R}^k} g(x)e^{-2i\pi(x+z)\cdot\xi}\mathrm{d}x\right) \\
&= \int_{\mathbb{R}^k} g(x)\cos[2\pi(x+z)\cdot\xi]\mathrm{d}x \\
&= 1 - \int_{\mathbb{R}^k} g(x)(1-\cos[2\pi(x+z)\cdot\xi])\mathrm{d}x.
\end{aligned}
$$

Our goal is to establish a positive lower bound on

$$\int_{\mathbb{R}^k} g(x)(1-\cos[2\pi(x+z)\cdot\xi])\mathrm{d}x.$$

The key point is that $(1-\cos[2\pi(x+z)\cdot\xi])$ is strictly positive except on a set of measure zero, and that since $\|g\|_p < \infty$, g cannot be too heavily concentrated on this set.

To proceed, we first localize. By the bound on the second moment and Markov's inequality, $\int_{|x-\mu|\le R} g(x)\mathrm{d}x \ge (1-\sigma^2/R^2)$. Choosing $R=\sqrt{2}\sigma$, we see that half of the mass, at least, is contained in the ball of radius $\sqrt{2}\sigma$, centered on μ.

Let $\tau \in (0, 1/2)$, to be chosen later. Define

$$B := \{x : |x-\mu| \le \sqrt{2}\sigma,\ 1-\cos[2\pi(x+z)\xi] \le \tau\}.$$

Then, since the integrand is non-negative,

$$
\begin{aligned}
&\int_{|x-\mu|\le\sqrt{2}\sigma} g(x)(1-\cos[2\pi(x+z)\xi])\,\mathrm{d}x \\
&\quad\ge \int_{\{|x-\mu|\le\sqrt{2}\sigma\}\setminus B} g(x)(1-\cos[2\pi(x+z)\xi])\,\mathrm{d}x. \\
&\quad\ge \tau \int_{\{|x-\mu|\le\sqrt{2}\sigma\}\setminus B} g(x)\mathrm{d}x.
\end{aligned}
\tag{9.3}
$$

Next, with $|B|$ denoting the Lebesgue measure of B, we have that $\int_B g(x)\mathrm{d}x \le \|g\|_\infty |B|$. Therefore,

$$\int_{\{|x|\le R\}\setminus B} g(x)\mathrm{d}x \ge 1/2 - \|g\|_\infty |B|. \tag{9.4}$$

To estimate $|B|$, note that if x lies in B, then $|x-\mu| \le \sqrt{2}\sigma$, and there exists $n \in \mathbb{Z}$ such that

$$\left|(x+z)\cdot\frac{\xi}{|\xi|} - \frac{n}{|\xi|}\right| \le \frac{\cos^{-1}(1-\tau)}{2\pi|\xi|}.$$

The points x that satisfy this inequality lie in parallel slabs of thickness $\cos^{-1}(1-\tau)/(\pi|\xi|)$, repeated at intervals of $1/|\xi|$. The intersection of any of these slabs with any ball of radius $\sqrt{2}\sigma$ has measure at most

$$\omega_{k-1}\sigma^{k-1}2^{(k-1)/2}\frac{\cos^{-1}(1-\tau)}{\pi|\xi|},$$

where ω_{k-1} denotes the volume of the unit ball in $\mathbb{R}^{k-1}$ for $k \geq 1$, and is 1 for $k = 1$. Also, for $\tau \leq 1/2$,

$$\cos^{-1}(1-\tau) \leq \sqrt{3\tau}.$$

Finally, since there are at most $2\sqrt{2}\sigma|\xi| + 3$ of the slabs that intersect the ball of radius $\sqrt{2}$, and since $\eta \leq 1$, $|B|$ can be estimated as follows, using the hypotheses $\eta \leq 1$ and $\sigma \geq 1$:

$$|B| \leq \omega_{k-1}\sigma^{k-1}2^{(k-1)/2}\left(\frac{2\sqrt{2}\sigma|\xi|+3}{\pi|\xi|}\right)\sqrt{3\tau} \leq \sigma^k\omega_{k-1}2^{(k-1)/2}\left(\frac{2\sqrt{2}+3}{\pi|\eta|}\right)\sqrt{3\tau}. \quad (9.5)$$

It is easy to see from (9.5) that one can choose τ to be a sufficiently small multiple of η^2, depending only on σ and K so that

$$\int_B g(x)\mathrm{d}x \leq 1/4.$$

Then from (9.3) and (9.4), one has the bound claimed in the lemma. It is easy to compute an explicit form for $\alpha(\|g\|_\infty, \sigma)$ from (9.5). The result for $k = 3$ given in (9.2) is obtained this way, where we have estimate $2\sqrt{2} + 3 \leq 6$ to simplify the appearance of the bound. $\square$

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